2-3 The Remainder and Factor Theorems

Factor each polynomial completely using the given factor and long division.

1. \( x^3 + 2x^2 - 23x - 60; x + 4 \)

**SOLUTION:**

\[
\begin{array}{c|ccccc}
 & x^3 & +2x^2 & -23x & -60 \\
\hline
x + 4 & x^2 & -2x & -15 \\
\hline & -x^3 & -2x^2 & 23x & 60 \\
& \hline & 0 & 0 & 0 & 0
\end{array}
\]

So, \( x^3 + 2x^2 - 23x - 60 = (x + 4)(x^2 - 2x - 15) \).

Factoring the quadratic expression yields \( x^3 + 2x^2 - 23x - 60 = (x + 4)(x - 5)(x + 3) \).

2. \( x^3 + 2x^2 - 21x + 18; x - 3 \)

**SOLUTION:**

\[
\begin{array}{c|ccccc}
 & x^3 & +5x & -6 \\
\hline
x - 3 & x^2 & +2x & -21 & +18 \\
\hline & -x^3 & -2x^2 & 21x & -6 \\
& \hline & 0 & 0 & 0 & 0
\end{array}
\]

So, \( x^3 + 2x^2 - 21x + 18 = (x - 3)(x^2 + 5x - 6) \).

Factoring the quadratic expression yields \( x^3 + 2x^2 - 21x + 18 = (x - 3)(x + 6)(x - 1) \).
2-3 The Remainder and Factor Theorems

3. \( x^3 + 3x^2 - 18x - 40; x - 4 \)

**SOLUTION:**

\[
\begin{array}{c|ccccc}
  & x^2 + 7x + 10 \\
-4 & x^3 + 3x^2 - 18x - 40 \\
\hline
  & 3x^2 - 4x \\
  & 7x^2 - 18x \\
-7 & 7x^3 - 28x \\
\hline
  & 10x - 40 \\
-10 & -10x + 40 \\
\hline
  & 0 \\
\end{array}
\]

So, \( x^3 + 3x^2 - 18x - 40 = (x - 4)(x^2 + 7x + 10) \).

Factoring the quadratic expression yields \( x^3 + 3x^2 - 18x - 40 = (x - 4)(x + 2)(x + 5) \).

4. \( 4x^3 + 20x^2 - 8x - 96; x + 3 \)

**SOLUTION:**

\[
\begin{array}{c|ccccc}
  & 4x^2 + 8x - 32 \\
x + 3 & 4x^3 + 20x^2 - 8x - 96 \\
\hline
  & 4x^2 + 12x \\
  & 8x^2 - 8x \\
-8 & 8x^3 - 24x \\
\hline
  & -32x - 96 \\
-32 & -32x + 96 \\
\hline
  & 0 \\
\end{array}
\]

So, \( 4x^3 + 20x^2 - 8x - 96 = (x + 3)(4x^2 + 8x - 32) \).

Factoring the quadratic expression yields \( 4x^3 + 20x^2 - 8x - 96 = 4(x + 3)(x + 4)(x - 2) \).

5. \( -3x^3 + 15x^2 + 108x - 540; x - 6 \)

**SOLUTION:**

\[
\begin{array}{c|ccccc}
  & -3x^2 - 3x + 90 \\
x - 6 & -3x^3 + 15x^2 + 108x - 540 \\
\hline
  & -3x^2 + 18x \\
  & -3x^2 + 108x \\
-90 & -3x^2 + 18x \\
\hline
  & 90x - 540 \\
90 & 90x - 540 \\
\hline
  & 0 \\
\end{array}
\]

So, \( -3x^3 + 15x^2 + 108x - 540 = (x - 6)(-3x^2 - 3x + 90) \).

Factoring the quadratic expression yields \( -3x^3 + 15x^2 + 108x - 540 = -3(x - 6)(x + 6)(x - 5) \).
6. \(6x^3 - 7x^2 - 29x - 12; 3x + 4\)

**SOLUTION:**

\[
\begin{array}{c|ccccc}
& 2x^2 & -5x & -3 \\
\hline
3x + 4 | 6x^3 & -7x^2 & -29x & -12 \\
\hline
& 6x^3 & +8x^2 & \\
\hline
& -15x^2 & -29x & \\
\hline
& -15x^2 & -20x & \\
\hline
& & -9x & -12 & \\
\hline
& & -9x & -12 & \\
\hline
& & & 0 & \\
\end{array}
\]

So, \(6x^3 - 7x^2 - 29x - 12 = (3x + 4)(2x^2 - 5x - 3)\).
Factoring the quadratic expression yields \(6x^3 - 7x^2 - 29x - 12 = (3x + 4)(2x + 1)(x - 3)\).

7. \(x^4 + 12x^3 + 38x^2 + 12x - 63; x^2 + 6x + 9\)

**SOLUTION:**

\[
\begin{array}{c|cccc}
& x^2 & +6x & -7 \\
\hline
x^2 + 6x + 9 | x^4 & +12x^3 & +38x^2 & +12x & -63 \\
\hline
& x^4 & +6x^3 & +9x^2 & \\
\hline
& 6x^3 & +29x^2 & +12x & \\
\hline
& 6x^3 & +36x^2 & +54x & \\
\hline
& -7x^2 & -42x & -63 & \\
\hline
& -7x^2 & -42x & -63 & \\
\hline
& & & 0 & \\
\end{array}
\]

So, \(x^4 + 12x^3 + 38x^2 + 12x - 63 = (x^2 + 6x + 9)(x^2 + 6x - 7)\).
Factoring both quadratic expressions yield \(x^4 + 12x^3 + 38x^2 + 12x - 63 = (x + 3)^2(x + 7)(x - 1)\).

8. \(x^4 - 3x^3 - 36x^2 + 68x + 240; x^2 - 4x - 12\)

**SOLUTION:**

\[
\begin{array}{c|ccccc}
& x^2 & +x & -20 \\
\hline
x^2 - 4x - 12 | x^4 & -3x^3 & -36x^2 & +68x & +240 \\
\hline
& x^4 & -4x^3 & -12x^2 & \\
\hline
& x^3 & -24x^2 & +68x & \\
\hline
& x^3 & -4x^2 & -12x & \\
\hline
& & -20x^2 & +80x & +240 & \\
\hline
& & -20x^2 & +80x & +240 & \\
\hline
& & & 0 & \\
\end{array}
\]

So, \(x^4 - 3x^3 - 36x^2 + 68x + 240 = (x^2 - 4x - 12)(x^2 + x - 20)\).
Factoring both quadratic expressions yield \(x^4 + 12x^3 + 38x^2 + 12x - 63 = (x - 6)(x + 2)(x + 5)(x - 4)\).
2-3 The Remainder and Factor Theorems

Divide using long division.

9. \((5x^4 - 3x^3 + 6x^2 - x + 12) ÷ (x - 4)\)

**SOLUTION:**

\[
\begin{array}{c|cccc}
\multicolumn{1}{r}{5x^3 + 17x^2 + 74x + 295} \\
\hline
x - 4 & 5x^4 - 3x^3 + 6x^2 - x + 12 \\
\hline
& 5x^4 - 20x^3 \\
& \underline{-9x^3 + 6x^2} \\
& \phantom{-}17x^3 + 26x^2 \\
& \underline{-17x^3 - 118x} \\
& \phantom{-}296x \\
& \underline{-296x + 1180} \\
& \phantom{-}1192 \\
\end{array}
\]

So, \(\frac{5x^4 + 3x^3 + 6x^2 + 12}{x - 4} = 5x^3 + 17x^2 + 74x + 295 + \frac{1192}{x - 4}\).

10. \((x^6 - 2x^5 + x^4 - x^3 + 3x^2 - x + 24) ÷ (x + 2)\)

**SOLUTION:**

\[
\begin{array}{c|cccc}
\multicolumn{1}{r}{x^5 - 4x^4 + 9x^3 - 19x^2 + 41x - 83} \\
\hline
x + 2 & x^6 - 2x^5 + x^4 - x^3 + 3x^2 - x + 24 \\
\hline
& x^5 + 2x^4 \\
& \underline{-4x^4 + x^3} \\
& \phantom{-}5x^4 - x^3 \\
& \underline{-5x^4 + 18x^3} \\
& \phantom{-}19x^3 + 3x^2 \\
& \underline{-19x^3 - 38x^2} \\
& \phantom{-}41x^2 - x \\
& \underline{-41x^2 + 82x} \\
& \phantom{-}83x + 24 \\
& \underline{-83x - 166} \\
& \phantom{-}190 \\
\end{array}
\]

So, \(\frac{x^6 + 2x^5 + x^4 + 3x^2 + 24}{x + 2} = x^5 - 4x^4 + 9x^3 - 19x^2 + 41x - 83 + \frac{190}{x + 2}\).
2-3 The Remainder and Factor Theorems

11. \((4x^4 - 8x^3 + 12x^2 - 6x + 12) ÷ (2x + 4)\)

\textit{SOLUTION:}

\[
\begin{array}{c|ccccc}
2x + 4 & 2x^3 - 8x^2 + 22x - 47 \\
\hline
\text{-}(4x^4 - 8x^3) & -16x^3 + 12x^2 \\
\text{-}(4x^4 - 8x^3) & 44x^2 - 6x \\
\text{-}(44x^2 + 88x) & -94x + 12 \\
\text{-}(-94x - 188) & 200 \\
\end{array}
\]

The remainder \(\frac{200}{2x + 4}\) can be written as \(\frac{100}{x + 2}\). So, \(\frac{4x^4 + 48x^3 + 12x^2 + 26x + 12}{2x + 4} = 2x^3 - 8x^2 + 22x - 47 + \frac{100}{x + 2}\).

12. \((2x^4 - 7x^3 - 38x^2 + 103x + 60) ÷ (x - 3)\)

\textit{SOLUTION:}

\[
\begin{array}{c|ccccc}
x - 3 & 2x^3 - x^2 - 41x - 20 \\
\hline
\text{-}(2x^4 - 6x^3) & -x^3 - 38x^2 \\
\text{-}(-x^3 - 38x^2) & -41x^2 + 103x \\
\text{-}(-41x^2 - 123x) & -20x + 60 \\
\text{-}(-20x - 60) & 0 \\
\end{array}
\]

So, \(\frac{2x^4 + 338x^2 + 103x + 60}{x - 3} = 2x^3 - x^2 - 41x - 20\).
2-3 The Remainder and Factor Theorems

13. \((6x^6 - 3x^5 + 6x^4 - 15x^3 + 2x^2 + 10x - 6) \div (2x - 1)\)

\textit{SOLUTION:}

\[
\begin{array}{c|cccc}
2x - 1 & 6x^6 - 3x^5 + 6x^4 - 15x^3 + 2x^2 + 10x - 6 \\
\hline
& 3x^5 + 3x^3 - 6x^2 - 2x + 4 \\
\vspace{-0.5cm}
\end{array}
\]

\[
\begin{array}{c|cccc}
2x - 1 & 6x^6 - 3x^5 + 6x^4 - 15x^3 + 2x^2 + 10x - 6 \\
& 3x^5 + 3x^3 - 6x^2 - 2x + 4 \\
\hline
& 0x^4 + 6x^3 - 15x^2 \\
\vspace{-0.5cm}
\end{array}
\]

\[
\begin{array}{c|cccc}
2x - 1 & 6x^6 - 3x^5 + 6x^4 - 15x^3 + 2x^2 + 10x - 6 \\
& 0x^4 + 6x^3 - 15x^2 \\
\hline
& -12x^3 + 2x^2 \\
\vspace{-0.5cm}
\end{array}
\]

\[
\begin{array}{c|cccc}
2x - 1 & 6x^6 - 3x^5 + 6x^4 - 15x^3 + 2x^2 + 10x - 6 \\
& -12x^3 + 2x^2 \\
\hline
& -4x^3 + 10x \\
\vspace{-0.5cm}
\end{array}
\]

\[
\begin{array}{c|cccc}
2x - 1 & 6x^6 - 3x^5 + 6x^4 - 15x^3 + 2x^2 + 10x - 6 \\
& -4x^3 + 2x \\
\hline
& 8x - 6 \\
\vspace{-0.5cm}
\end{array}
\]

\[
\begin{array}{c|cccc}
2x - 1 & 6x^6 - 3x^5 + 6x^4 - 15x^3 + 2x^2 + 10x - 6 \\
& 8x - 4 \\
\hline
& -2 \\
\vspace{-0.5cm}
\end{array}
\]

So, \(\frac{6x^6 - 3x^5 + 6x^4 - 15x^3 + 2x^2 + 10x - 6}{2x - 1} = 3x^4 + 3x^3 - 6x^2 - 2x + 4 - \frac{2}{2x - 1}\)

14. \((108x^5 - 36x^4 + 75x^2 + 36x + 24) \div (3x + 2)\)

\textit{SOLUTION:}

\[
\begin{array}{c|cccc}
3x + 2 & 108x^5 - 36x^4 + 0x^3 + 75x^2 + 36x + 24 \\
\hline
& 36x^4 - 36x^3 + 24x^2 + 9x + 6 \\
\vspace{-0.5cm}
\end{array}
\]

\[
\begin{array}{c|cccc}
3x + 2 & 108x^5 - 36x^4 + 0x^3 + 75x^2 + 36x + 24 \\
& 36x^4 - 36x^3 + 24x^2 + 9x + 6 \\
\hline
& -108x^4 + 0x^3 \\
\vspace{-0.5cm}
\end{array}
\]

\[
\begin{array}{c|cccc}
3x + 2 & 108x^5 - 36x^4 + 0x^3 + 75x^2 + 36x + 24 \\
& -108x^4 + 0x^3 \\
\hline
& 72x^3 + 75x^2 \\
\vspace{-0.5cm}
\end{array}
\]

\[
\begin{array}{c|cccc}
3x + 2 & 108x^5 - 36x^4 + 0x^3 + 75x^2 + 36x + 24 \\
& 72x^3 + 75x^2 \\
\hline
& -72x^3 + 48x^2 \\
\vspace{-0.5cm}
\end{array}
\]

\[
\begin{array}{c|cccc}
3x + 2 & 108x^5 - 36x^4 + 0x^3 + 75x^2 + 36x + 24 \\
& -72x^3 + 48x^2 \\
\hline
& 27x^2 + 36x \\
\vspace{-0.5cm}
\end{array}
\]

\[
\begin{array}{c|cccc}
3x + 2 & 108x^5 - 36x^4 + 0x^3 + 75x^2 + 36x + 24 \\
& 27x^2 + 36x \\
\hline
& 18x + 24 \\
\vspace{-0.5cm}
\end{array}
\]

\[
\begin{array}{c|cccc}
3x + 2 & 108x^5 - 36x^4 + 0x^3 + 75x^2 + 36x + 24 \\
& 18x + 24 \\
\hline
& 18x + 12 \\
\vspace{-0.5cm}
\end{array}
\]

\[
\begin{array}{c|cccc}
3x + 2 & 108x^5 - 36x^4 + 0x^3 + 75x^2 + 36x + 24 \\
& 18x + 12 \\
\hline
& 12 \\
\vspace{-0.5cm}
\end{array}
\]

So, \(\frac{108x^5 - 36x^4 + 75x^2 + 36x + 24}{3x + 2} = 36x^4 - 36x^3 + 24x^2 + 9x + 6 + \frac{12}{3x + 2}\)
2-3 The Remainder and Factor Theorems

15. \((x^4 + x^3 + 6x^2 + 18x - 216) ÷ (x^3 - 3x^2 + 18x - 54)\)

**SOLUTION:**

\[
\begin{array}{c|ccccc}
& x^4 & + & x^3 & + & 6x^2 & + 18x - 216 \\
\hline
x^3 - 3x^2 + 18x - 54) & x & + & 4 \\
\hline
& x^4 & - & 3x^3 & - & 18x^2 & - 54x \\
\hline
& 4x^3 & - & 12x^2 & - & 72x & - 216 \\
\hline
& - & 4x^3 & - & 12x^2 & - & 72x & - 216 \\
\hline
& & & & & 0 \\
\hline
\end{array}
\]

So, \(\frac{x^4 + x^3 + 6x^2 + 18x - 216}{x^3 - 3x^2 + 18x - 54} = x + 4.\)

16. \((4x^4 - 14x^3 - 14x^2 + 110x - 84) ÷ (2x^2 + x - 12)\)

**SOLUTION:**

\[
\begin{array}{c|cccc}
& 2x^2 & - & 8x & + & 9 \\
\hline
2x^2 + x - 12 & 4x^4 & - & 14x^3 & - & 14x^2 & + 110x & - 84 \\
\hline
(-) 4x^4 & + & 2x^3 & - & 24x^2 \\
\hline
& - & 16x^3 & + & 10x^2 & + 110x \\
\hline
(-) -16x^3 & - & 8x^2 & + & 96x \\
\hline
& 18x^2 & + & 14x & - & 84 \\
\hline
(-) 18x^2 & + & 9x & - 108 \\
\hline
& & & & 5x & + 24 \\
\hline
\end{array}
\]

So, \(\frac{4x^4 + 14x^3 + 314x^2 + 110x - 84}{2x^2 + x - 12} = 2x^2 - 8x + 9 + \frac{5x + 24}{2x^2 + x - 12}.\)

17. \(\frac{6x^5 - 12x^4 + 10x^3 - 2x^2 - 8x + 8}{3x^3 + 2x + 3}\)

**SOLUTION:**

\[
\begin{array}{c|cccc}
& 2x^2 & - & 4x & + & 2 \\
\hline
3x^3 + 2x + 3 & 6x^5 & - & 12x^4 & + 10x^3 & - 2x^2 & - 8x & + 8 \\
\hline
(-) 6x^5 & + & 4x^4 & + & 6x^2 \\
\hline
& -12x^4 & + & 6x^3 & - 8x^2 & - 8x \\
\hline
(-) -12x^4 & - & 8x^2 & - & 12x \\
\hline
& 6x^3 & + & 4x & + 8 \\
\hline
(-) 6x^3 & + & 4x & + 6 \\
\hline
& & & & 2 \\
\hline
\end{array}
\]

So, \(\frac{6x^5 - 12x^4 + 10x^3 - 2x^2 - 8x + 8}{3x^3 + 2x + 3} = 2x^2 - 4x + 2 + \frac{2}{3x^3 + 2x + 3}.\)
2-3 The Remainder and Factor Theorems

18. \( \frac{12x^5 + 5x^4 - 15x^3 + 19x^2 - 4x - 28}{3x^3 + 2x^2 + 6} \)

**SOLUTION:**

\[
\begin{array}{c|cccccc}
& 4x^2 - x - 3 \\
3x^3 + 2x^2 - x + 6 & 12x^5 + 5x^4 - 15x^3 + 19x^2 - 4x - 28 \\
\hline
& (-) 12x^5 + 8x^4 - 4x^3 + 24x^2 \\
& -3x^4 - 11x^3 - 5x^2 - 4x \\
& (-) -3x^4 - 2x^3 + x^2 - 6x \\
& -9x^3 - 6x^2 + 2x - 28 \\
& (-) -9x^3 - 6x^2 + 3x - 18 \\
& -x - 10
\end{array}
\]

The remainder \( \frac{-x - 10}{3x^3 + 2x^2 - x + 6} \) can be written as \( \frac{x + 10}{3x^3 + 2x^2 - x + 6} \). So,

\[
\frac{12x^5 + 5x^4 - 15x^3 + 19x^2 - 4x - 28}{3x^3 + 2x^2 + 6} = 4x^2 - x - 3 - \frac{x + 10}{3x^3 + 2x^2 - x + 6}
\]

**Divide using synthetic division.**

19. \( (x^4 - x^3 + 3x^2 - 6x - 6) \div (x - 2) \)

**SOLUTION:**

Because \( x = 2 \), \( c = 2 \). Set up the synthetic division as follows. Then follow the synthetic division procedure.

\[
\begin{array}{c|cccc}
2 & 1 & -1 & 3 & -6 & -6 \\
\hline
& 2 & 2 & 10 & 8 \\
1 & 5 & 4 & 2 & 2
\end{array}
\]

The quotient is \( x^3 + x^2 + 5x + 4 + \frac{2}{x - 2} \).

20. \( (2x^4 + 4x^3 - 2x^2 + 8x - 4) \div (x + 3) \)

**SOLUTION:**

Because \( x = -3 \), \( c = -3 \). Set up the synthetic division as follows. Then follow the synthetic division procedure.

\[
\begin{array}{c|cccc}
-3 & 2 & 4 & -2 & 8 & -4 \\
\hline
& -6 & 6 & -12 & 12 \\
2 & -2 & 4 & -4 & 8
\end{array}
\]

The quotient is \( 2x^3 - 2x^2 + 4x - 4 + \frac{8}{x + 3} \).
2-3 The Remainder and Factor Theorems

21. \((3x^4 - 9x^3 - 24x - 48) \div (x - 4)\)

**SOLUTION:**
Because \(x - 4, c = 4\). Set up the synthetic division as follows, using a zero placeholder for the missing \(x^2\)-term in the dividend. Then follow the synthetic division procedure.

\[
\begin{array}{c|cccc}
4 & 3 & -9 & 0 & -24 & -48 \\
--- & --- & --- & --- & --- & --- \\
 & 12 & 12 & 48 & 96 \\
\end{array}
\]

The quotient is \(3x^3 + 3x^2 + 12x + 24 + \frac{48}{x - 4}\).

22. \((x^5 - 3x^3 + 6x^2 + 9x + 6) \div (x + 2)\)

**SOLUTION:**
Because \(x + 2, c = -2\). Set up the synthetic division as follows, using a zero placeholder for the missing \(x^4\)-term in the dividend. Then follow the synthetic division procedure.

\[
\begin{array}{c|ccccc}
-2 & 1 & 0 & -3 & 6 & 9 & 6 \\
--- & --- & --- & --- & --- & --- & --- \\
 & -2 & 4 & -2 & -8 & -2 \\
\end{array}
\]

The quotient is \(x^4 - 2x^3 + x^2 + 4x + 1 + \frac{4}{x + 2}\).

23. \((12x^5 + 10x^4 - 18x^3 - 12x^2 - 8) \div (2x - 3)\)

**SOLUTION:**
Rewrite the division expression so that the divisor is of the form \(x - c\).

\[
\frac{12x^5 + 10x^4 - 18x^3 - 12x^2 - 8}{2x - 3} = \frac{(12x^5 + 10x^4 - 18x^3 - 12x^2 - 8) + 2}{(2x - 3) + 2} = \frac{6x^5 + 5x^4 - 9x^3 - 6x^2 - 4}{x - \frac{3}{2}}.
\]

Because \(x - \frac{3}{2}, c = \frac{3}{2}\). Set up the synthetic division as follows, using a zero placeholder for the missing \(x\)-term in the dividend. Then follow the synthetic division procedure.

\[
\begin{array}{c|ccccc}
\frac{3}{2} & 6 & 5 & -9 & -6 & 0 & -4 \\
--- & --- & --- & --- & --- & --- & --- \\
 & 9 & 21 & 18 & 18 & 27 \\
\end{array}
\]

The remainder \(\frac{23}{x - \frac{3}{2}}\) can be written as \(\frac{46}{2x - 3}\). So, the quotient is \(6x^4 + 14x^3 + 12x^2 + 12x + 18 + \frac{46}{2x - 3}\).
2-3 The Remainder and Factor Theorems

24. \((36x^4 - 6x^3 + 12x^2 - 30x - 12) \div (3x + 1)\)

**SOLUTION:**
Rewrite the division expression so that the divisor is of the form \(x - c\).
\[
\frac{36x^4 - 6x^3 + 12x^2 - 30x - 12}{3x + 1} = \frac{(36x^4 - 6x^3 + 12x^2 - 30x - 12) \div 3}{(3x + 1) \div 3}
\]
\[
= \frac{12x^4 - 2x^3 + 4x^2 - 10x - 4}{x + \frac{1}{3}}
\]
Because \(x + \frac{1}{3}, c = -\frac{1}{3}\): Set up the synthetic division as follows. Then follow the synthetic division procedure.

\[
\begin{array}{c|cccc}
-\frac{1}{3} & 12 & -2 & 4 & -10 & -4 \\
\hline
& -4 & 2 & -2 & 4 & \\
& 12 & -6 & 6 & -12 & |0
\end{array}
\]
The quotient is \(12x^3 - 6x^2 + 6x - 12\).

25. \((45x^5 + 6x^4 + 3x^3 + 8x + 12) \div (3x - 2)\)

**SOLUTION:**
Rewrite the division expression so that the divisor is of the form \(x - c\).
\[
\frac{45x^5 + 6x^4 + 3x^3 + 8x + 12}{3x - 2} = \frac{(45x^5 + 6x^4 + 3x^3 + 8x + 12) \div 3}{(3x - 2) \div 3}
\]
\[
= \frac{15x^5 + 2x^4 + x^3 + \frac{8}{3}x + 4}{x - \frac{2}{3}}
\]
Because \(x - \frac{2}{3}, c = \frac{2}{3}\): Set up the synthetic division as follows, using a zero placeholder for the missing \(x^2\)-term in the dividend. Then follow the synthetic division procedure.

\[
\begin{array}{c|ccccc}
\frac{2}{3} & 15 & 2 & 1 & 0 & \frac{8}{3} & 4 \\
\hline
& 10 & 8 & 6 & 4 & 40 & 9 \\
& 15 & 12 & 9 & 6 & 20 & \frac{76}{3} & \frac{76}{9}
\end{array}
\]
The remainder \(\frac{76}{9} \), can be written as \(\frac{76}{3(3x - 2)}\). So, the quotient is \(15x^4 + 12x^3 + 9x^2 + 6x + \frac{20}{3} + \frac{76}{3(3x - 2)}\).
2-3 The Remainder and Factor Theorems

26. \((48x^5 + 28x^4 + 68x^3 + 11x + 6) ÷ (4x + 1)\)

**SOLUTION:**
Rewrite the division expression so that the divisor is of the form \(x - c\).

\[
\frac{48x^5 + 28x^4 + 68x^3 + 11x + 6}{4x + 1} = \frac{48x^5 + 28x^4 + 68x^3 + 11x + 6}{4} \div 4
\]

\[
= \frac{12x^5 + 7x^4 + 17x^3 + \frac{11}{4}x + \frac{3}{2}}{x + \frac{1}{4}}
\]

Because \(x + \frac{1}{4}, c = -\frac{1}{4}\). Set up the synthetic division as follows, using a zero placeholder for the missing \(x^2\)-term in the dividend. Then follow the synthetic division procedure.

\[
\begin{array}{ccccccc}
\frac{1}{4} & | & 12 & 7 & 17 & 0 & 11 & 3 \\
\end{array}
\]

\[
\begin{array}{ccccccc}
& & -3 & -1 & -4 & 1 & -15 \\
\end{array}
\]

\[
\begin{array}{ccccccc}
12 & 4 & 16 & -4 & 15 & 4 & 9 \\
\end{array}
\]

The remainder \(\frac{9}{16}\) can be written as \(\frac{9}{4(4x + 1)}\). So, the quotient is \(12x^4 + 4x^3 + 16x^2 - 4x + \frac{15}{4} + \frac{9}{4(4x + 1)}\).

27. \((60x^6 + 78x^5 + 9x^4 - 12x^3 - 25x - 20) ÷ (5x + 4)\)

**SOLUTION:**
Rewrite the division expression so that the divisor is of the form \(x - c\).

\[
\frac{60x^6 + 78x^5 + 9x^4 - 12x^3 - 25x - 20}{5x + 4} = \frac{60x^6 + 78x^5 + 9x^4 - 12x^3 - 25x - 20}{5} ÷ 5
\]

\[
= \frac{12x^6 + \frac{78}{5}x^5 + \frac{9}{5}x^4 - \frac{12}{5}x^3 - 5x - 4}{x + \frac{4}{5}}
\]

Because \(x + \frac{4}{5}, c = -\frac{4}{5}\). Set up the synthetic division as follows, using a zero placeholder for the missing \(x^2\)-term in the dividend. Then follow the synthetic division procedure.

\[
\begin{array}{ccccccc}
\frac{4}{5} & | & 12 & \frac{78}{5} & \frac{9}{5} & -\frac{12}{5} & 0 & -5 & -4 \\
\end{array}
\]

\[
\begin{array}{ccccccc}
& & -48 & -24 & 12 & 0 & 0 & 4 \\
\end{array}
\]

\[
\begin{array}{ccccccc}
12 & 6 & -3 & 0 & 0 & -5 & 0 \\
\end{array}
\]

The quotient is \(12x^5 + 6x^4 - 3x^3 - 5\).
2-3 The Remainder and Factor Theorems

28. \[
\frac{16x^6 - 56x^5 - 24x^4 + 96x^3 - 42x^2 - 30x + 105}{2x - 7}
\]

**SOLUTION:**
Rewrite the division expression so that the divisor is of the form \(x - c\).
\[
\frac{16x^6 - 56x^5 - 24x^4 + 96x^3 - 42x^2 - 30x + 105}{2x - 7} = \frac{(16x^6 - 56x^5 - 24x^4 + 96x^3 - 42x^2 - 30x + 105) + 2(2x - 7)^2}{2x - 7}
\]
\[
= \frac{8x^5 - 28x^4 - 12x^3 + 48x^2 - 21x^2 - 15x + \frac{105}{2}}{x - \frac{7}{2}}
\]

Because \(x - \frac{7}{2}\), \(c = \frac{7}{2}\). Set up the synthetic division as follows, using a zero placeholder for the missing \(x^2\)-term in the dividend. Then follow the synthetic division procedure.

\[
\begin{array}{cccccc}
& 7 & 2 & 8 & -28 & -12 & 48 & -21 & -15 & \frac{105}{2} \\
\hline
\frac{7}{2} & & & 28 & 0 & -42 & 21 & 0 & -\frac{105}{2} & \\
\hline
& 8 & 0 & -12 & 6 & 0 & -15 & 0 & &
\end{array}
\]

The quotient is \(8x^5 - 12x^4 + 6x^2 - 15\).

29. **EDUCATION** The number of U.S. students, in thousands, that graduated with a bachelor’s degree from 1970 to 2006 can be modeled by \(g(x) = 0.0002x^5 - 0.016x^4 + 0.512x^3 - 7.15x^2 + 47.52x + 800.27\), where \(x\) is the number of years since 1970. Use synthetic substitution to find the number of students that graduated in 2005. Round to the nearest thousand.

**SOLUTION:**
To find the number of students that graduated in 2005, use synthetic substitution to evaluate \(g(x)\) for \(x = 35\).

\[
\begin{array}{cccccc}
35 & 0.0002 & -0.016 & 0.512 & -7.15 & 47.52 & 800.27 \\
\hline
0.0002 & -0.099 & 0.197 & -0.255 & 38.595 & 2151.095 \\
\hline
\end{array}
\]

The remainder is 2151.095, so \(g(35) = 2151.095\). Therefore, rounded to the nearest thousand, 2,151,000 students graduated in 2005.

30. **SKIING** The distance in meters that a person travels on skis can be modeled by \(d(t) = 0.2t^2 + 3t\), where \(t\) is the time in seconds. Use the Remainder Theorem to find the distance traveled after 45 seconds.

**SOLUTION:**
To find the distance traveled after 45 seconds, use synthetic substitution to evaluate \(d(t)\) for \(t = 45\).

\[
\begin{array}{cccc}
45 & 0.2 & 3 & 0 \\
\hline
9 & 540 \\
\hline
0.2 & 12 & 540 \\
\hline
\end{array}
\]

The remainder is 540, so \(d(45) = 540\). Therefore, 540 meters were traveled in 45 seconds.
2-3 The Remainder and Factor Theorems

Find each \( f(c) \) using synthetic substitution.

31. \( f(x) = 4x^5 - 3x^4 + x^3 - 6x^2 + 8x - 15; c = 3 \)

**SOLUTION:**

\[
\begin{array}{c|cccccc}
3 & 4 & -3 & 1 & -6 & 8 & -15 \\
 & & 12 & 27 & 84 & 234 & 726 \\
\hline
 & 4 & 9 & 28 & 78 & 242 & 711 \\
\end{array}
\]

The remainder is 711. Therefore, \( f(3) = 711 \).

32. \( f(x) = 3x^6 - 2x^5 + 4x^4 - 2x^3 + 8x - 3; c = 4 \)

**SOLUTION:**

\[
\begin{array}{c|ccccc}
4 & 3 & -2 & 4 & -2 & 0 & 8 & -3 \\
 & & 12 & 40 & 176 & 696 & 2784 & 11168 \\
\hline
 & 3 & 10 & 44 & 174 & 696 & 2792 & 11165 \\
\end{array}
\]

The remainder is 11,165. Therefore, \( f(4) = 11,165 \).

33. \( f(x) = 2x^6 + 5x^5 - 3x^4 + 6x^3 - 9x^2 + 3x - 4; c = 5 \)

**SOLUTION:**

\[
\begin{array}{c|ccccc}
5 & 2 & 5 & -3 & 6 & -9 & 3 & -4 \\
 & & 10 & 75 & 360 & 1830 & 9105 & 45540 \\
\hline
 & 2 & 15 & 72 & 366 & 1821 & 9108 & 45536 \\
\end{array}
\]

The remainder is 45,536. Therefore, \( f(5) = 45,536 \).

34. \( f(x) = 4x^6 + 8x^5 - 6x^4 - 5x^2 + 6x - 4; c = 6 \)

**SOLUTION:**

\[
\begin{array}{c|cccccc}
6 & 4 & 8 & 0 & -6 & -5 & 6 & -4 \\
 & & 24 & 192 & 1152 & 6876 & 41226 & 247392 \\
\hline
 & 4 & 32 & 192 & 1146 & 6871 & 41232 & 247388 \\
\end{array}
\]

The remainder is 247,388. Therefore, \( f(6) = 247,388 \).

35. \( f(x) = 10x^5 + 6x^4 - 8x^3 + 7x^2 - 3x + 8; c = -6 \)

**SOLUTION:**

\[
\begin{array}{c|cccccc}
-6 & 10 & 6 & -8 & 7 & -3 & 8 \\
 & & -60 & 324 & -1896 & 11334 & -67986 \\
\hline
 & 10 & -54 & 316 & -1889 & 11331 & -67978 \\
\end{array}
\]

The remainder is -67,978. Therefore, \( f(-6) = -67,978 \).
2-3 The Remainder and Factor Theorems

36. \( f(x) = -6x^7 + 4x^5 - 8x^4 + 12x^3 - 15x^2 - 9x + 64; c = 2 \)

**SOLUTION:**

\[
\begin{array}{c|cccccccc}
  2 & -6 & 0 & 4 & -8 & 12 & -15 & -9 & 64 \\
  \hline
  & -6 & -12 & -20 & -48 & -84 & -183 & -375 & -686 \\
\end{array}
\]

The remainder is \(-686\). Therefore, \(f(2) = -686\).

37. \( f(x) = -2x^8 + 6x^5 - 4x^4 + 12x^3 - 6x + 24; c = 4 \)

**SOLUTION:**

\[
\begin{array}{c|cccccccc}
  4 & -2 & 0 & 0 & 6 & -4 & 12 & 0 & -6 & 24 \\
  \hline
\end{array}
\]

The remainder is \(-125,184\). Therefore, \(f(4) = -125,184\).

Use the Factor Theorem to determine if the binomials given are factors of \(f(x)\). Use the binomials that are factors to write a factored form of \(f(x)\).

38. \( f(x) = x^4 - 2x^3 - 9x^2 + x + 6; (x + 2), (x - 1) \)

**SOLUTION:**

Use synthetic division to test each factor, \((x + 2)\) and \((x - 1)\).

\[
\begin{array}{c|ccccc}
  -2 & 1 & -2 & -9 & 1 & 6 \\
    & 1 & -4 & -1 & 3 & |0 \\
  \hline
  & 1 & 8 & 2 & -6 & |0 \\
\end{array}
\]

Because the remainder when \(f(x)\) is divided by \((x + 2)\) is 0, \((x + 2)\) is a factor. Test the second factor, \((x - 1)\), with the depressed polynomial \(x^3 - 4x^2 - x + 3\).

\[
\begin{array}{c|cccc}
  1 & 1 & -4 & -1 & 3 \\
    & 1 & -3 & -4 & |0 \\
  \hline
  & 1 & -3 & -4 & -1 & |0 \\
\end{array}
\]

Because the remainder when the depressed polynomial is divided by \((x - 1)\) is \(-1\), \((x - 1)\) is not a factor of \(f(x)\). Because \((x + 2)\) is a factor of \(f(x)\), we can use the quotient of \(f(x) \div (x + 2)\) to write a factored form of \(f(x)\) as \(f(x) = (x + 2)(x^3 - 4x^2 - x + 3)\).

39. \( f(x) = x^4 + 2x^3 - 5x^2 + 8x + 12; (x - 1), (x + 3) \)

**SOLUTION:**

Use synthetic division to test each factor, \((x - 1)\) and \((x + 3)\).

\[
\begin{array}{c|ccccccc}
  1 & 2 & -5 & 8 & 12 & -3 & 1 & 2 & -5 & 8 & 12 \\
    & 1 & 3 & -2 & 6 & -3 & 3 & 6 & -42 & |0 \\
  \hline
  & 1 & 3 & -2 & 6 & 18 & 1 & -1 & -2 & 14 & -30 \\
\end{array}
\]

Because the remainder when \(f(x)\) is divided by \((x - 1)\) is 18, \((x - 1)\) is not a factor. Because the remainder when \(f(x)\) is divided by \((x + 3)\) is \(-30\), \((x + 3)\) is not a factor.
2-3 The Remainder and Factor Theorems

40. \( f(x) = x^4 - 2x^3 + 24x^2 + 18x + 135; \ (x - 5), \ (x + 5) \)

**SOLUTION:**

Use synthetic division to test each factor, \((x - 5)\) and \((x + 5)\).

\[
\begin{array}{c|cccc}
5 & 1 & -2 & 24 & 18 & 135 \\
 & 5 & 15 & 195 & 1065 \\
\hline
1 & 3 & 39 & 213 & 1200
\end{array}
\]

Because the remainder when \(f(x)\) is divided by \((x - 5)\) is 1200, \((x - 5)\) is not a factor.

\[
\begin{array}{c|cccc}
-5 & 1 & -2 & 24 & 18 & 135 \\
 & -5 & 35 & -295 & 1385 \\
\hline
1 & -7 & 59 & -277 & 1520
\end{array}
\]

Because the remainder when \(f(x)\) is divided by \((x + 5)\) is 1520, \((x + 5)\) is not a factor.

41. \( f(x) = 3x^4 - 22x^3 + 13x^2 + 118x - 40; \ (3x - 1), \ (x - 5) \)

**SOLUTION:**

Use synthetic division to test each factor, \((3x - 1)\) and \((x - 5)\).
For \((3x - 1)\), rewrite the division expression so that the divisor is of the form \(x - c\).

\[
\frac{3x^4 - 22x^3 + 13x^2 + 118x - 40}{3x - 1} = \frac{(3x^4 - 22x^3 + 13x^2 + 118x - 40) \div 3}{(3x - 1) \div 3}
\]

\[
\frac{x^4 - \frac{22}{3}x^3 + \frac{13}{3}x^2 + \frac{118}{3}x - \frac{40}{3}}{x - \frac{1}{3}}
\]

Because \(x - \frac{1}{3}, c = \frac{1}{3}\), Set up the synthetic division as follows. Then follow the synthetic division procedure.

\[
\begin{array}{c|cccc}
\frac{1}{3} & 1 & -\frac{22}{3} & \frac{13}{3} & \frac{118}{3} & -40 \\
 & 1 & -\frac{7}{3} & 2 & 40 \\
\hline
1 & -\frac{7}{3} & 2 & 40 & 0
\end{array}
\]

Because the remainder when \(f(x)\) is divided by \((3x - 1)\) is 0, \((3x - 1)\) is a factor. Test the second factor, \((x - 5)\), with the depressed polynomial \(x^3 - 7x^2 + 2x + 40\).

\[
\begin{array}{c|cccc}
5 & 1 & -7 & 2 & 40 \\
 & 5 & -10 & -40 \\
\hline
1 & -2 & -8 & 0
\end{array}
\]

Because the remainder when the depressed polynomial is divided by \((x - 5)\) is 0, \((x - 5)\) is a factor of \(f(x)\).

Because \((3x - 1)\) and \((x - 5)\) are factors of \(f(x)\), we can use the final quotient to write a factored form of \(f(x)\) as \(f(x) = (3x - 1)(x - 5)(x^2 - 2x - 8)\). Factoring the quadratic expression yields \(f(x) = (3x - 1)(x - 5)(x - 4)(x + 2)\).
2-3 The Remainder and Factor Theorems

42. \(f(x) = 4x^4 - x^3 - 36x^2 - 111x + 30; (4x - 1), (x - 6)\)

\textbf{SOLUTION:}

Use synthetic division to test each factor, \((4x - 1)\) and \((x - 6)\).

For \((4x - 1)\), rewrite the division expression so that the divisor is of the form \(x - c\).

\[
\frac{4x^4 - x^3 - 36x^2 - 111x + 30}{4x - 1} = \frac{(4x^4 - x^3 - 36x^2 - 111x + 30) \div 4}{(4x - 1) \div 4}
\]

\[
= \frac{x^4 - \frac{1}{4}x^3 - 9x^2 - \frac{111}{4}x + \frac{30}{4}}{x - \frac{1}{4}}
\]

Because \(x - \frac{1}{4}, c = \frac{1}{4}\). Set up the synthetic division as follows. Then follow the synthetic division procedure.

\[
\begin{array}{c|cccc}
1 & 4 & -1 & -9 & -111 & 30 \\
\hline
4 & 4 & -9 & -4 & -4 & 4 \\
1 & 0 & -9 & -30 & 0 \\
\end{array}
\]

Because the remainder when \(f(x)\) is divided by \((4x - 1)\) is 0, \((4x - 1)\) is a factor. Test the second factor, \((x - 6)\), with the depressed polynomial \(x^3 - 9x - 30\).

\[
\begin{array}{c|cccc}
6 & 1 & 0 & -9 & -30 \\
\hline
6 & 6 & 36 & 162 \\
1 & 6 & 27 & 132 \\
\end{array}
\]

Because the remainder when the depressed polynomial is divided by \((x - 6)\) is 132, \((x - 6)\) is not a factor of \(f(x)\).

Because \((4x - 1)\) is a factor of \(f(x)\), we can use the quotient of \(f(x) \div (4x - 1)\) to write a factored form of \(f(x)\) as \(f(x) = (4x - 1)(x^3 - 9x - 30)\).
2-3 The Remainder and Factor Theorems

43. \( f(x) = 3x^4 - 35x^3 + 38x^2 + 56x + 64; (3x - 2), (x + 2) \)

\textit{SOLUTION:}

Use synthetic division to test each factor, \((3x - 2)\) and \((x + 2)\).

For \((3x - 2)\), rewrite the division expression so that the divisor is of the form \(x - c\).

\[
\frac{3x^4 - 35x^3 + 38x^2 + 56x + 64}{3x - 2} = \frac{(3x^4 - 35x^3 + 38x^2 + 56x + 64) \div 3}{(3x - 2) \div 3}
\]

\[
= \frac{x^4 - \frac{35}{3}x^3 + \frac{38}{3}x^2 + \frac{56}{3}x + \frac{64}{3}}{x - \frac{2}{3}}
\]

Because \(x - \frac{2}{3}, c = \frac{2}{3}\). Set up the synthetic division as follows. Then follow the synthetic division procedure.

\[
\begin{array}{c|cccc}
2 & 35 & 38 & 56 & 64 \\
3 & & -2 & & \\
2 & 22 & 32 & 400 \\
3 & 3 & 9 & 27 \\
\hline
1 & -11 & 16 & 200 & 976 \\
\end{array}
\]

Because the remainder when \(f(x)\) is divided by \((3x - 2)\) is \(\frac{976}{27}\), \((3x - 2)\) is not a factor.

Test \((x + 2)\).

\[
\begin{array}{c|cccc}
-2 & 3 & -35 & 38 & 56 & 64 \\
3 & & -6 & 82 & -240 & 368 \\
\hline
3 & -41 & 120 & -184 & 432 \\
\end{array}
\]

Because the remainder when \(f(x)\) is divided by \((x + 2)\) is \(432\), \((x + 2)\) is not a factor.
2-3 The Remainder and Factor Theorems

44. \( f(x) = 5x^5 + 38x^4 - 68x^2 + 59x + 30; (5x - 2), (x + 8) \)

**SOLUTION:**

Use synthetic division to test each factor, \((5x - 2)\) and \((x + 8)\).

For \((5x - 2)\), rewrite the division expression so that the divisor is of the form \(x - c\).

\[
\frac{5x^5 + 38x^4 - 68x^2 + 59x + 30}{5x - 2} = \frac{(5x^5 + 38x^4 - 68x^2 + 59x + 30) \div 5}{(5x - 2) \div 5}
\]

\[
= \frac{x^5 + \frac{38}{5} x^4 - \frac{68}{5} x^2 + \frac{59}{5} x + 6}{x - \frac{2}{5}}
\]

Because \(x - \frac{2}{5}, c = \frac{2}{5}\). Set up the synthetic division as follows. Then follow the synthetic division procedure.

\[
\begin{array}{c|cccccc}
2 & \frac{38}{5} & 0 & \frac{68}{5} & \frac{59}{5} & 6 \\
\hline
5 & 2 & 16 & 32 & 616 & 1718 \\
& 5 & 5 & 25 & 125 & 625 \\
\hline
1 & 8 & 16 & 308 & 859 & 5468 \\
& 5 & 25 & 125 & 625 & \\
\end{array}
\]

Because the remainder when \(f(x)\) is divided by \((5x - 2)\) is \(\frac{5468}{625}\), \((5x - 2)\) is not a factor.

Test \((x + 8)\).

\[
\begin{array}{c|ccccc}
-8 & 5 & 38 & 0 & -68 & 59 & 30 \\
\hline
& -40 & 16 & -128 & 1568 & -13016 \\
\hline
5 & -2 & 16 & -196 & 1627 & -12986 \\
\end{array}
\]

Because the remainder when \(f(x)\) is divided by \((x + 8)\) is \(-12986\), \((x + 8)\) is not a factor.
2-3 The Remainder and Factor Theorems

45. \( f(x) = 4x^5 - 9x^4 + 39x^3 + 24x^2 + 75x + 63; (4x + 3), (x - 1) \)

**SOLUTION:**

Use synthetic division to test each factor, \((4x + 3)\) and \((x - 1)\).

For \((4x + 3)\), rewrite the division expression so that the divisor is of the form \(x - c\).

\[
\frac{4x^5 - 9x^4 + 39x^3 + 24x^2 + 75x + 63}{4x + 3} = \frac{(4x^5 - 9x^4 + 39x^3 + 24x^2 + 75x + 63) ÷ 4}{(4x + 3) ÷ 4}
\]

\[
\frac{x^5 - \frac{9}{4}x^4 + \frac{39}{4}x^3 + 6x^2 + \frac{75}{4}x + \frac{63}{4}}{x + \frac{3}{4}}
\]

Because \(x + \frac{3}{4}, c = -\frac{3}{4}\). Set up the synthetic division as follows. Then follow the synthetic division procedure.

\[
\begin{array}{c|cccccc}
-\frac{3}{4} & 1 & -9 & 43 & 6 & 75 & 63 \\
\hline
4 & 4 & 4 & 4 & 4 & 4 \\
\hline
3 & 9 & -9 & 9 & 63 \\
\hline
4 & 4 & 4 & 4 & 4 \\
\hline
1 & -3 & 12 & -3 & 21 & 0
\end{array}
\]

Because the remainder when \(f(x)\) is divided by \((4x + 3)\) is 0, \((4x + 3)\) is a factor. Test the second factor, \((x - 1)\), with the depressed polynomial \(x^4 - 3x^3 + 12x^2 - 3x + 21\).

\[
\begin{array}{c|cccc}
1 & 1 & -3 & 12 & -3 \\
\hline
1 & 0 & -2 & 10 & 7 \\
\hline
1 & -2 & 10 & 7 & 28
\end{array}
\]

Because the remainder when the depressed polynomial is divided by \((x - 1)\) is 28, \((x - 1)\) is not a factor of \(f(x)\). Because \((4x + 3)\) is a factor of \(f(x)\), we can use the quotient of \(f(x) ÷ (4x + 3)\) to write a factored form of \(f(x)\) as \(f(x) = (4x + 3)(x^4 - 3x^3 + 12x^2 - 3x + 21)\).
2-3 The Remainder and Factor Theorems

46. **TREES** The height of a tree in feet at various ages in years is given in the table.

<table>
<thead>
<tr>
<th>Age</th>
<th>Height</th>
<th>Age</th>
<th>Height</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>3.3</td>
<td>24</td>
<td>73.8</td>
</tr>
<tr>
<td>6</td>
<td>13.8</td>
<td>26</td>
<td>82.0</td>
</tr>
<tr>
<td>10</td>
<td>25.0</td>
<td>28</td>
<td>91.9</td>
</tr>
<tr>
<td>14</td>
<td>42.7</td>
<td>30</td>
<td>101.7</td>
</tr>
<tr>
<td>20</td>
<td>60.7</td>
<td>36</td>
<td>111.5</td>
</tr>
</tbody>
</table>

a. Use a graphing calculator to write a quadratic equation to model the growth of the tree.
b. Use synthetic division to evaluate the height of the tree at 15 years.

**SOLUTION:**

a. Use the quadratic regression function on the graphing calculator.

\[
y = -0.001x^2 + 3.44x - 6.39
\]

b. To find the height of the tree at 15 years, use synthetic substitution to evaluate \( f(x) \) for \( x = 15 \).

\[
\begin{array}{ccc}
15 & -0.001 & 3.44 & -6.39 \\
 & -0.015 & 51.375 \\
 & -0.001 & 3.425 & 44.985 \\
\end{array}
\]

The remainder is 44.985, so \( f(15) = 44.985 \). Therefore, the height of the tree at 15 years is about 44.985 feet.
2-3 The Remainder and Factor Theorems

47. **BICYCLING** Patrick is cycling at an initial speed $v_0$ of 4 meters per second. When he rides downhill, the bike accelerates at a rate $a$ of 0.4 meter per second squared. The vertical distance from the top of the hill to the bottom of the hill is 25 meters. Use $d(t) = v_0t + \frac{1}{2}at^2$ to find how long it will take Patrick to ride down the hill, where $d(t)$ is distance traveled and $t$ is given in seconds.

**SOLUTION:**

Substitute $v_0 = 4$, $a = 0.4$, and $d(t) = 25$ into $d(t) = v_0t + \frac{1}{2}at^2$.

\[
d(t) = v_0t + \frac{1}{2}at^2
\]
\[
25 = (4)t + \frac{1}{2}(0.4)t^2
\]
\[
25 = 4t + 0.2t^2
\]
\[
0 = 0.2t^2 + 4t - 25
\]

Use the quadratic equation to solve for $t$.

\[
\frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-4 \pm \sqrt{4^2 - 4(0.2)(-25)}}{2(0.2)}
\]
\[
= \frac{-4 \pm 6}{0.4}
\]
\[
= 5 \text{ or } -25
\]

It will take Patrick 5 seconds to travel the 25 meters.

Factor each polynomial using the given factor and long division. Assume $n > 0$.

48. $x^{3n} + x^{2n} - 14x^n - 24; x^n + 2$

**SOLUTION:**

\[
\begin{array}{c|c}
\multicolumn{2}{c}{x^n + 2} \\
\hline
x^n & x^{3n} + x^{2n} - 14x^n - 24 \\
(\text{-}) & x^{3n} + 2x^{2n} \\
& -x^{2n} - 14x^n \\
(\text{-}) & -x^{2n} - 2x^n \\
& -12x^n - 24 \\
(\text{-}) & -12x^n - 24 \\
& 0
\end{array}
\]

So, $x^{3n} + x^{2n} - 14x^n - 24 = (x^n + 2)(x^{2n} - x^n - 12)$.

Factoring the quadratic expression yields $x^{3n} + x^{2n} - 14x^n - 24 = (x^n + 2)(x^n - 4)(x^n + 3)$.
2-3 The Remainder and Factor Theorems

49. \(x^{3n} + x^{2n} - 12x^n + 10; x^n - 1\)

**SOLUTION:**

\[
x^n - 1 \big| x^{3n} + x^{2n} - 12x^n + 10
\]

\[
(-) x^{3n} - x^{2n} \\
\quad 2x^{2n} - 12x^n \\
(-) 2x^{2n} - 2x^n \\
\quad -10x^n + 10 \\
(-) -10x^n + 10 \\
\quad 0
\]

So, \(x^{3n} + x^{2n} - 12x^n + 10 = (x^n - 1)(x^{2n} + 2x^n - 10)\).

50. \(4x^{3n} + 2x^{2n} - 10x^n + 4; 2x^n + 4\)

**SOLUTION:**

\[
2x^n + 4 \div 4x^{3n} + 2x^{2n} - 10x^n + 4
\]

\[
(-) 4x^{3n} + 8x^{2n} \\
\quad -6x^{2n} - 10x^n \\
(-) -6x^{2n} - 12x^n \\
\quad 2x^n + 4 \\
(-) 2x^n + 4 \\
\quad 0
\]

So, \(4x^{3n} + 2x^{2n} - 10x^n + 4 = (2x^n + 4)(2x^{2n} - 3x^n + 1)\).

Factoring the quadratic expression yields \(4x^{3n} + 2x^{2n} - 10x^n + 4 = (2x^n + 4)(2x^n - 1)(x^n - 1)\).

51. \(9x^{3n} + 24x^{2n} - 171x^n + 54; 3x^n - 1\)

**SOLUTION:**

\[
3x^n - 1 \big| 9x^{3n} + 24x^{2n} - 171x^n + 54
\]

\[
(-) 9x^{3n} - 3x^{2n} \\
\quad 27x^{2n} - 171x^n \\
(-) 27x^{2n} - 9x^n \\
\quad -162x^n + 54 \\
(-) -162x^n + 54 \\
\quad 0
\]

So, \(9x^{3n} + 24x^{2n} - 171x^n + 54 = (3x^n - 1)(3x^{2n} + 9x^n - 54)\).

Factoring the quadratic expression yields \(9x^{3n} + 24x^{2n} - 171x^n + 54 = 3(3x^n - 1)(x^n + 6)(x^n - 3)\).

52. MANUFACTURING An 18-inch by 20-inch sheet of cardboard is cut and folded into a bakery box.
2-3 The Remainder and Factor Theorems

a. Write a polynomial function to model the volume of the box.
b. Graph the function.
c. The company wants the box to have a volume of 196 cubic inches. Write an equation to model this situation.
d. Find a positive integer for \( x \) that satisfies the equation found in part c.

**SOLUTION:**

a. The length of the box is \( 18 - 2x \). The height is \( x \). The width of the box is \( \frac{20 - 3x}{2} \). To find the volume, calculate the product.

\[
\begin{align*}
v(x) &= (18 - 2x) \left( \frac{20 - 3x}{2} \right) (x) \\
&= (18 - 2x) \left( 10 - \frac{3}{2} x \right) (x) \\
&= (180 - 27x - 20x + 3x^2) x \\
&= (3x^3 - 47x^2 + 180)x \\
&= 3x^3 - 47x^2 + 180x
\end{align*}
\]

b. Evaluate the function for several \( x \)-values in its domain. The height, width, and length of the box must all be positive values. For the height, \( x > 0 \). For the length, \( 18 - 2x > 0 \) or \( x < 9 \). For the width, \( \frac{20 - 3x}{2} > 0 \) or \( x < \frac{20}{3} \).

Thus, the domain of \( x \) is \( 0 < x < \frac{20}{3} \).

<table>
<thead>
<tr>
<th>( x )</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
</tr>
</thead>
<tbody>
<tr>
<td>( v(x) )</td>
<td>0</td>
<td>136</td>
<td>196</td>
<td>198</td>
<td>160</td>
<td>100</td>
<td>36</td>
</tr>
</tbody>
</table>

Use these points to construct a graph.

c. Substitute \( v(x) = 196 \) into the original equation to arrive at \( 196 = 3x^3 - 47x^2 + 180x \).

d. Using the trace function on a graphing calculator, it appears that \( v(x) \) may be 196 in\(^3\) when \( x = 2 \).
2-3 The Remainder and Factor Theorems

If \( x = 2 \) is a solution for the equation, it will also be a solution to \( 0 = 3x^3 - 47x^2 + 180x - 196 \). Use synthetic substitution to verify that \( x = 2 \) is a solution.

\[
\begin{array}{c|ccc}
2 & 3 & -47 & 180 & -196 \\
& & 6 & -82 & 196 \\
\hline
& 3 & -41 & 98 & 0
\end{array}
\]

Because the remainder is 0, \((x - 2)\) is a factor of \(3x^3 - 47x^2 + 180x - 196\). Thus, \( x = 2 \) is a solution to \( 196 = 3x^3 - 47x^2 + 180x \).

**Find the value of \( k \) so that each remainder is zero.**

53. \( \frac{x^3k+2x+4}{x} \)

**SOLUTION:**

Because \( x - 2 \), \( c = 2 \). Set up the synthetic division as follows. Then follow the synthetic division procedure.

\[
\begin{array}{c|ccc}
2 & 1 & -k & 2 & -4 \\
& & 2 & 4-2k & 12-4k \\
\hline
& 1 & -k & 2-6k & 8-4k
\end{array}
\]

The remainder is \( 8 - 4k \). Solve \( 8 - 4k = 0 \) for \( k \).

\[
8 - 4k = 0 \\
-4k = -8 \\
k = 2
\]

When \( k = 2 \), \( \frac{x^3k+2x+4}{x} \) will have a remainder of 0.
2-3 The Remainder and Factor Theorems

54. \( \frac{x^3+18x^2+kx+4}{x+2} \)

**SOLUTION:**
Because \( x + 2 \), \( c = -2 \). Set up the synthetic division as follows. Then follow the synthetic division procedure.

\[
\begin{array}{c|ccccc}
-2 & 1 & 18 & k & 4 \\
\hline
 & 1 & 16 & k-32 & \underline{-2k+64} \\
\end{array}
\]

The remainder is \(-2k+68\). Solve \(-2k + 68 = 0\) for \(k\).
\(-2k + 68 = 0\)
\[-2k = -68\]
\[k = 34\]

When \(k = 34\), \( \frac{x^3+18x^2+kx+4}{x+2} \) will have a remainder of 0.

55. \( \frac{x^3+4x^2kx+1}{x+1} \)

**SOLUTION:**
Because \( x + 1 \), \( c = -1 \). Set up the synthetic division as follows. Then follow the synthetic division procedure.

\[
\begin{array}{c|ccc}
-1 & 1 & 4 & -k & 1 \\
\hline
 & 1 & 3 & k-3 & \underline{k+4} \\
\end{array}
\]

The remainder is \(k + 4\). Solve \(k + 4 = 0\) for \(k\).
\(k + 4 = 0\)
\[k = -4\]

When \(k = 4\), \( \frac{x^3+4x^2kx+1}{x+1} \) will have a remainder of 0.

56. \( \frac{2x^3x^2+x+k}{x+1} \)

**SOLUTION:**
Because \( x - 1 \), \( c = 1 \). Set up the synthetic division as follows. Then follow the synthetic division procedure.

\[
\begin{array}{c|ccc}
1 & 2 & -1 & 1 & k \\
\hline
 & 2 & 1 & \underline{2} & \underline{k+2} \\
\end{array}
\]

The remainder is \(k + 2\). Solve \(k + 2 = 0\) for \(k\).
\(k + 2 = 0\)
\[k = -2\]

When \(k = -2\), \( \frac{2x^3x^2+x+k}{x+1} \) will have a remainder of 0.

57. **SCULTPING** Esteban will use a block of clay that is 3 feet by 4 feet by 5 feet to make a sculpture. He wants to reduce the volume of the clay by removing the same amount from the length, the width, and the height.
2-3 The Remainder and Factor Theorems

a. Write a polynomial function to model the situation.
b. Graph the function.
c. He wants to reduce the volume of the clay to \( \frac{3}{5} \) of the original volume. Write an equation to model the situation.
d. How much should he take from each dimension?

**SOLUTION:**
a. Let the length of the block equal 3 feet, the width of the block equal 4 feet, and the height of the block equal 5 feet. Let \( x \) equal the amount removed. Thus, the length is \( 3 - x \), the width is \( 4 - x \), and the height is \( 5 - x \). The volume of the block is the product of the length, width, and height.

\[
v = hwh
\]
\[
v(x) = (3-x)(4-x)(5-x)
\]
\[
= (3-x)(20-9x+x^2)
\]
\[
= (60 - 27x + 3x^2 - 20x + 9x^2 - x^3)
\]
\[
= -x^3 + 12x^2 - 47x + 60
\]
A polynomial function to model the situation is \( v(x) = -x^3 + 12x^2 - 47x + 60 \).

b. Evaluate the function for several \( x \)-values in its domain. The length, width, and height of the box must all be positive values. For the length, \( 3 - x > 0 \) or \( x < 3 \). For the width, \( 4 - x > 0 \) or \( x < 4 \). For the width, \( 5 - x > 0 \) or \( x < 5 \). Thus, the domain of \( x \) is \( 0 < x < 3 \).

<table>
<thead>
<tr>
<th>( x )</th>
<th>0</th>
<th>0.5</th>
<th>1</th>
<th>1.5</th>
<th>2</th>
<th>2.5</th>
</tr>
</thead>
<tbody>
<tr>
<td>( v(x) )</td>
<td>60</td>
<td>39.4</td>
<td>24</td>
<td>13.1</td>
<td>6</td>
<td>1.9</td>
</tr>
</tbody>
</table>

Use these points to construct a graph.

c. The volume of the original block of clay is \( 3 \cdot 4 \cdot 5 = 60 \) cubic feet. If Esteban reduces the volume by \( \frac{3}{5} \), the volume of the new block will be \( 60 \cdot \frac{3}{5} = 36 \) cubic feet. An equation to model the situation is \( 36 = -x^3 + 12x^2 - 47x + 60 \).
d. To solve the equation \( 36 = -x^3 + 12x^2 - 47x + 60 \) for \( x \), use a graphing calculator to graph \( y = 36 \) and \( y = -x^3 + 12x^2 - 47x + 60 \) on the same screen. Use the intersect function from the **CALC** menu to find \( x \).
2-3 The Remainder and Factor Theorems

Use the graphs and synthetic division to completely factor each polynomial.

58. \( f(x) = 8x^4 + 26x^3 - 103x^2 - 156x + 45 \)

\[ \begin{array}{c|cccc}
-5 & 8 & 26 & -103 & -156 & 45 \\
 & & -40 & 70 & 165 & -45 \\
\hline
 & 8 & -14 & -33 & 9 & 0
\end{array} \]

Because the remainder when \( f(x) \) is divided by \((x + 5)\) is 0, \((x + 5)\) is a factor. Test the second factor, \((x - 3)\), with the depressed polynomial \( 8x^3 - 14x^2 - 33x + 9 \).

\[ \begin{array}{c|ccc}
3 & 8 & -14 & -33 & 9 \\
 & 24 & 30 & -9 \\
\hline
 & 8 & 10 & -3 & 0
\end{array} \]

Because the remainder when the depressed polynomial is divided by \((x - 3)\) is 0, \((x - 3)\) is a factor of \( f(x) \). Because \((x + 5)\) and \((x - 3)\) are factors of \( f(x) \), we can use the final quotient to write a factored form of \( f(x) \) as \( f(x) = (x + 5)(x - 3)(8x^2 + 10x - 3) \). Factoring the quadratic expression yields \( f(x) = (x + 5)(x - 3)(4x - 1)(2x + 3) \).
2-3 The Remainder and Factor Theorems

59. \( f(x) = 6x^5 + 13x^4 - 153x^3 + 54x^2 + 724x - 840 \)

\[ \begin{array}{cccccc}
\text{s} & 6 & 13 & -153 & 54 & 724 & -840 \\
\text{r} & & -36 & 138 & 90 & -864 & 840 \\
\text{d} & 6 & -23 & -15 & 144 & -140 & \text{r} = 0 \\
\\hline
\\end{array} \]

Because the remainder when \( f(x) \) is divided by \( x + 6 \) is 0, \( x + 6 \) is a factor. Test the second factor, \( x - 2 \), with the depressed polynomial \( 6x^4 - 23x^3 - 15x^2 + 144x - 140 \).

\[ \begin{array}{cccccc}
2 & 6 & -23 & -15 & 144 & -140 \\
\text{s} & 12 & -22 & -74 & -140 & \text{s} = 0 \\
\\hline
\\end{array} \]

Because the remainder when the depressed polynomial is divided by \( x - 2 \) is 0, \( x - 2 \) is a factor of \( f(x) \).

The depressed polynomial is \( 6x^3 - 11x^2 - 37x + 70 \). To find factors of this polynomial, use a graphing calculator to observe the graph.

The graph suggests that \( x - 2 \) may be a factor of the depressed polynomial. Use synthetic division to test the factor \( x - 2 \).

\[ \begin{array}{cccccc}
2 & 6 & -11 & -37 & 70 & \text{s} = 0 \\
\\hline
\\end{array} \]

Because the remainder when the depressed polynomial is divided by \( x - 2 \) is 0, \( x - 2 \) is a repeated factor of \( f(x) \). Because \( x + 6 \) and \( x - 2 \) are factors of \( f(x) \), we can use the final quotient to write a factored form of \( f(x) \) as \( f(x) = (x + 6)(x - 2)^2(6x^2 + x - 35) \). Factoring the quadratic expression yields \( f(x) = (x + 6)(x - 2)(3x - 7)(2x + 5) \).

60. **MULTIPLE REPRESENTATIONS** In this problem, you will explore the upper and lower bounds of a function.

**a. GRAPHICAL** Graph each related polynomial function and determine the greatest and least zeros. Then copy and complete the table.
2-3 The Remainder and Factor Theorems

b. NUMERICAL Use synthetic division to evaluate each function in part a for three integer values greater than the greatest zero.

c. VERBAL Make a conjecture about the characteristics of the last row when synthetic division is used to evaluate a function for an integer greater than its greatest zero.

d. NUMERICAL Use synthetic division to evaluate each function in part a for three integer values less than the least zero.

e. VERBAL Make a conjecture about the characteristics of the last row when synthetic division is used to evaluate a function for a number less than its least zero.

SOLUTION:

a. Use a graphing calculator to graph \( y = x^3 - 2x^2 - 11x + 12 \).

The least zero appears to \(-3\). The greatest zero appears to be 4.

Use a graphing calculator to graph \( y = x^4 + 6x^3 + 3x^2 - 10x \).

The least zero appears to \(-5\). The greatest zero appears to be 1.

Use a graphing calculator to graph \( y = x^5 - x^4 - 2x^3 \).

The least zero appears to \(-1\). The greatest zero appears to be 2.
Factor each polynomial completely using the given factor and long division.

1. $x^3 + 2x^2 – 23x – 60$; $x + 4$.

2. The Remainder and Factor Theorems

b. Sample answer: For $x^3 - 2x^2 - 11x + 12$, use synthetic division for $c = 5, 7, $ and $8$.

2-3 The Remainder and Factor Theorems

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2-3 The Remainder and Factor Theorems

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2-3 The Remainder and Factor Theorems

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2-3 The Remainder and Factor Theorems

b. Sample answer: For $x^3 - 2x^2 - 11x + 12$, use synthetic division for $c = 5, 7, $ and $8$.
2-3 The Remainder and Factor Theorems

\[ \begin{array}{c|cccc} -6 & 1 & -2 & -11 & 12 \\ \hline & 1 & 6 & 3 & -10 \\ & -6 & 0 & -18 & 168 \\ \hline 1 & 0 & 3 & -28 & 168 \\ \hline -7 & 1 & 6 & 3 & -10 \\ \hline & -7 & 7 & -70 & 560 \\ \hline & 1 & -1 & 10 & -80 \\ & -9 & 27 & -270 & 2520 \\ \hline & 1 & -3 & 30 & -280 \\ \end{array} \]

\[ f(-4) = -40, f(-5) = -108, f(-6) = -210 \]

For \( x^4 + 6x^3 + 3x^2 - 10 \), use synthetic division for \( c = 2, 3, \) and 4.

\[ \begin{array}{c|cccc} -6 & 1 & 6 & 3 & -10 \\ \hline & 1 & 6 & 3 & -10 \\ \hline & -6 & 0 & -18 & 168 \\ \hline 1 & 0 & 3 & -28 & 168 \\ \hline -7 & 1 & 6 & 3 & -10 \\ \hline & -7 & 7 & -70 & 560 \\ \hline & 1 & -1 & 10 & -80 \\ & -9 & 27 & -270 & 2520 \\ \hline & 1 & -3 & 30 & -280 \\ \end{array} \]

\[ f(-6) = 168, f(-7) = 560, f(-9) = 2520 \]

For \( x^5 - x^4 - 2x^3 \), use synthetic division for \( c = 3, 5, \) and 7.

\[ \begin{array}{c|cccc} -2 & 1 & -1 & -2 & 0 \\ \hline & 1 & -1 & -2 & 0 \\ \hline & -2 & 6 & -8 & 16 \\ \hline 1 & -3 & 4 & -8 & 16 \\ \hline -3 & 1 & -1 & -2 & 0 \\ \hline & -3 & 12 & -30 & 90 \\ \hline & 1 & -4 & 10 & -30 \\ \hline -4 & 1 & -1 & -2 & 0 \\ \hline & -4 & 20 & -72 & 288 \\ \hline & 1 & -5 & 18 & -72 \\ \end{array} \]

\[ f(-2) = -32, f(-3) = -270, f(-4) = -1152 \]

e. Sample answer: The elements in the last row alternate between nonnegative and nonpositive.

61. **CHALLENGE** Is \((x - 1)\) a factor of \(18x^{165} - 15x^{135} + 8x^{105} - 15x^{55} + 4\)? Explain your reasoning.

**SOLUTION:**

Yes; sample answer: Using the Factor Theorem, \((x - 1)\) is a factor if \(f(1) = 0\).

\[
\begin{align*}
f(1) &= 18(1)^{165} - 15(1)^{135} + 8(1)^{105} - 15(1)^{55} + 4 \\
&= 18(1) - 15(1) + 8(1) - 15(1) + 4 \\
&= 18 - 15 + 8 - 15 + 4 \\
&= 0
\end{align*}
\]

Since \(f(1) = 0\), \((x - 1)\) is a factor of the polynomial.
2-3 The Remainder and Factor Theorems

62. **Writing in Math**  Explain how you can use a graphing calculator, synthetic division, and factoring to completely factor a fifth-degree polynomial with rational coefficients, three integral zeros, and two non-integral, rational zeros.

**SOLUTION:**
Sample answer: I would use my graphing calculator to graph the polynomial and to determine the three integral zeros, a, b, and c. I would then use synthetic division to divide the polynomial by a. I would then divide the resulting depressed polynomial by b, and then the new depressed polynomial by c. The third depressed polynomial will have degree 2. Finally, I would factor the second-degree polynomial to find the two non-integral, rational zeros, d and e. So, the polynomial is either the product \((x - a)(x - b)(x - c)(x - d)(x - e)\) or this product multiplied by some rational number.

63. **REASONING**  Determine whether the statement below is true or false. Explain.

If \(h(y) = (y + 2)(3y^2 + 11y - 4) - 1\), then the remainder of \(\frac{h(y)}{y + 2}\) is -1.

**SOLUTION:**
True; sample answer: The Remainder Theorem states that if \(h(y)\) is divided by \(y - (-2)\), then the remainder is \(r = h(-2)\), which is -1.

**CHALLENGE**  Find \(k\) so that the quotient has a 0 remainder.

\[
\frac{x^3 + kx^2 - 34x + 56}{x + 7}
\]

**SOLUTION:**
Because \(x + 7, c = -7\). Set up the synthetic division as follows. Then follow the synthetic division procedure.

\[
\begin{array}{cccc}
-7 & 1 & k & -34 & 56 \\
& -7 & -7k + 49 & 49k - 105 \\
& 1 & k - 7 & -7k + 15 & 49k - 49
\end{array}
\]

The remainder is \(49k - 49\). Solve \(49k - 49 = 0\) for \(k\).

\[
49k = 49\Rightarrow k = 1
\]

When \(k = 49\), \(\frac{x^3 + kx^2 - 34x + 56}{x + 7}\) will have a remainder of 0.
2-3 The Remainder and Factor Theorems

65. \( \frac{x^6 + kx^4 - 8x^3 + 173x^2 - 16x - 120}{x - 1} \)

**SOLUTION:**
Because \( x - 1 \), \( c = 1 \). Set up the synthetic division as follows. Then follow the synthetic division procedure.

\[
\begin{array}{cccccc}
1 & 0 & -8 & 173 & -16 & -120 \\
1 & k & k + 1 & k - 7 & k + 166 & k + 150 \\
1 & 1 & k + 1 & k - 7 & k + 166 & k + 150 & k + 30
\end{array}
\]

The remainder is \( k + 30 \). Solve \( k + 30 = 0 \) for \( k \).

\( k + 30 = 0 \)

\( k = -30 \)

When \( k = -30 \), \( \frac{x^6 + kx^4 - 8x^3 + 173x^2 - 16x - 120}{x - 1} \) will have a remainder of 0.

66. \( \frac{kx^3 + 2x^2 - 22x - 4}{x - 2} \)

**SOLUTION:**
Because \( x - 2 \), \( c = 2 \). Set up the synthetic division as follows. Then follow the synthetic division procedure.

\[
\begin{array}{cccc}
2 & k & 2 & -22 & -4 \\
2k & 4k + 4 & 8k - 36 \\
k & 2k + 2 & 4k - 18 & 8k - 40
\end{array}
\]

The remainder is \( 8k - 40 \). Solve \( 8k - 40 = 0 \) for \( k \).

\( 8k - 40 = 0 \)

\( 8k = 40 \)

\( k = 5 \)

When \( k = 5 \), \( \frac{kx^3 + 2x^2 - 22x - 4}{x - 2} \) will have a remainder of 0.
2-3 The Remainder and Factor Theorems

67. CHALLENGE  If \( 2x^2 - dx + (31 - d^2) \) \( x + 5 \) has a factor \( x - d \), what is the value of \( d \) if \( d \) is an integer?

SOLUTION:

\( 2x^2 - dx + (31 - d^2) \) \( x + 5 \) can be written as \( 2x^2 + (-d^2 - d + 31)x + 5 \). Because \( (x - d) \) is a factor, \( c = d \). Set up the synthetic division as follows. Then follow the synthetic division procedure.

\[
\begin{array}{cccc|c}
2 & -d^2 - d + 31 & 5 \\
2d & -d^3 + d^2 + 31d \\
\hline
2 & -d^2 + d + 31 & -d^3 + d^2 + 31d + 5 \\
\end{array}
\]

The remainder is \(-d^3 + d^2 + 31d + 5\). Let \(-d^3 + d^2 + 31d + 5 = 0\) and solve for \( d \). Use a graphing calculator to graph \( y = -d^3 + d^2 + 31d + 5 \).

The graph suggests that possible values for \( d \) are \(-5\), \(0\), and \(6\). Use synthetic division to test each possible factor.

\[
\begin{array}{cccc|c}
-5 & 1 & 31 & 5 \\
\hline & 5 & -30 & -5 \\
\hline & -1 & 6 & 1 & 0 \\
\end{array}
\]

Since the remainder is 0, \( (x - (-5)) \) or \( (x + 5) \) is a factor. Test the two remaining values using the depressed polynomial.

\[
\begin{array}{cccc|c}
6 & -1 & 6 & 1 \\
\hline & 0 & 0 \\
\hline & -1 & 6 & 1 \\
\end{array}
\]

Since the remainder for both is 1, neither one is a factor. Thus, \( d = -5 \).

68. Writing in Math  Compare and contrast polynomial division using long division and using synthetic division.

SOLUTION:

Sample answer: Both long division and synthetic division can be used to divide a polynomial by a linear factor. Long division can also be used to divide a polynomial by a nonlinear factor. In synthetic division, only the coefficients are used. In both long division and synthetic division, placeholders are needed if a power of a variable is missing.
2-3 The Remainder and Factor Theorems

Determine whether the degree \( n \) of the polynomial for each graph is even or odd and whether its leading coefficient \( a_n \) is positive or negative.

69.

**SOLUTION:**
Since \( \lim_{x \to -\infty} f(x) = -\infty \) and \( \lim_{x \to \infty} f(x) = -\infty \), \( n \) is even and \( a_n \) is negative.

70.

**SOLUTION:**
Since \( \lim_{x \to -\infty} f(x) = -\infty \) and \( \lim_{x \to \infty} f(x) = \infty \), \( n \) is odd and \( a_n \) is positive.

71.

**SOLUTION:**
Since \( \lim_{x \to -\infty} f(x) = \infty \) and \( \lim_{x \to \infty} f(x) = -\infty \), \( n \) is odd and \( a_n \) is negative.
2-3 The Remainder and Factor Theorems

72. **SKYDIVING** The approximate time \( t \) in seconds that it takes an object to fall a distance of \( d \) feet is given by

\[
t = \frac{d}{\sqrt{16}}.
\]

Suppose a skydiver falls 11 seconds before the parachute opens. How far does the skydiver fall during this time period?

**SOLUTION:**

Substitute \( t = 11 \) into \( t = \frac{d}{\sqrt{16}} \) and solve for \( d \).

\[
t = \frac{d}{\sqrt{16}}
\]

\[
11 = \frac{d}{\sqrt{16}}
\]

\[
121 = \frac{d}{16}
\]

\[
1936 = d
\]

The skydiver falls 1936 feet.
2-3 The Remainder and Factor Theorems

73. **FIRE FIGHTING** The velocity \( v \) and maximum height \( h \) of water being pumped into the air are related by \( v = \sqrt{2gh} \), where \( g \) is the acceleration due to gravity (32 feet/second\(^2\)).

a. Determine an equation that will give the maximum height of the water as a function of its velocity.

b. The Mayfield Fire Department must purchase a pump that is powerful enough to propel water 80 feet into the air. Will a pump that is advertised to project water with a velocity of 75 feet/second meet the fire department’s needs? Explain.

**SOLUTION:**

a. Substitute \( g = 32 \) into \( v = \sqrt{2gh} \) and solve for \( h \).

\[
\begin{align*}
v &= \sqrt{2gh} \\
v &= \sqrt{2(32)h} \\
v &= \sqrt{64h} \\
v^2 &= 64h \\
\frac{v^2}{64} &= h
\end{align*}
\]

An equation that will give the maximum height of the water as a function of its velocity is \( h = \frac{v^2}{64} \).

b. Substitute \( v = 75 \) into the equation found in part a.

\[
\begin{align*}
\frac{v^2}{64} &= h \\
\frac{(75)^2}{64} &= h \\
87.89 &= h
\end{align*}
\]

The pump can propel water to a height of about 88 feet. So, the pump will meet the fire department’s needs.
2-3 The Remainder and Factor Theorems

Solve each system of equations algebraically.

74. \(5x - y = 16\)
\(2x + 3y = 3\)

**SOLUTION:**

\(5x - y = 16\) can be written as \(y = 5x - 16\). Substitute \(5x - 16\) for \(y\) into the second equation and solve for \(x\).
\[2x + 3(5x - 16) = 3\]
\[2x + 15x - 48 = 3\]
\[17x - 48 = 3\]
\[17x = 51\]
\[x = 3\]

Substitute \(x = 3\) into \(y = 5x - 16\) and solve for \(y\).
\[y = 5(3) - 16\]
\[y = 15 - 16\]
\[y = -1\]

The solution to the system of equations is \((3, -1)\).

75. \(3x - 5y = -8\)
\(x + 2y = 1\)

**SOLUTION:**

\(x + 2y = 1\) can be written as \(x = 1 - 2y\). Substitute \(1 - 2y\) for \(x\) into the first equation and solve for \(y\).
\[3x - 5y = -8\]
\[3(1 - 2y) - 5y = -8\]
\[3 - 6y - 5y = -8\]
\[3 - 11y = -8\]
\[-11y = -11\]
\[y = 1\]

Substitute \(y = 1\) into \(x = 1 - 2y\) and solve for \(x\).
\[x = 1 - 2(1)\]
\[x = 1 - 2\]
\[x = -1\]

The solution to the system of equations is \((-1, 1)\).
2-3 The Remainder and Factor Theorems

76. \( y = 6 - x \)
\( x = 4.5 + y \)

**SOLUTION:**
Substitute \( 6 - x \) for \( y \) into the second equation and solve for \( x \).
\[ x = 4.5 + y \]
\[ x = 4.5 + (6 - x) \]
\[ x = 10.5 - x \]
\[ 2x = 10.5 \]
\[ x = 5.25 \]

Substitute \( x = 5.25 \) into \( y = 6 - x \) and solve for \( y \).
\[ y = 6 - x \]
\[ y = 6 - 5.25 \]
\[ y = 0.75 \]

The solution to the system of equation is \((5.25, 0.75)\).

77. \( 2x + 5y = 4 \)
\( 3x + 6y = 5 \)

**SOLUTION:**
Eliminate \( x \).
\[ 2x + 5y = 4 \rightarrow (3x) \rightarrow 6x + 15y = 12 \]
\[ 3x + 6y = 5 \rightarrow (-2x) \rightarrow (+) - 6x - 12y = 10 \]
\[ 3y = 2 \]

Solve for \( y \).
\[ 3y = 2 \]
\[ y = \frac{2}{3} \]

Substitute \( y = \frac{2}{3} \) into the second equation and solve for \( x \).
\[ 3x + 6y = 5 \]
\[ 3x + 6 \left( \frac{2}{3} \right) = 5 \]
\[ 3x + 4 = 5 \]
\[ 3x = 1 \]
\[ x = \frac{1}{3} \]

The solution to the system of equation is \( \left( \frac{1}{3}, \frac{2}{3} \right) \).
2-3 The Remainder and Factor Theorems

78. $7x + 12y = 16$
   $5y - 4x = -21$

**SOLUTION:**

$5y - 4x = -21$ can be written as $-4x + 5y = -21$. Eliminate $x$.

$7x + 12y = 16 \rightarrow (4\times) \rightarrow 28x + 48y = 64$

$-4x + 5y = -21 \rightarrow (7\times) \rightarrow (+) - 28x + 35y = -147$

\[ 83y = -83 \]

Solve for $y$.

$83y = -83$

$y = -1$

Substitute $y = -1$ into the first equation and solve for $x$.

$7x + 12y = 16$

$7x + 12(-1) = 16$

$7x - 12 = 16$

$7x = 28$

$x = 4$

The solution to the system of equation is $(4, -1)$. 
2-3 The Remainder and Factor Theorems

79. 4x + 5y = 8
   3x - 7y = 10

   SOLUTION:
   Eliminate x.
   4x + 5y = 8 → (3x) → 12x + 15y = -24
   3x - 7y = 10 → (-4x) → (+) -12x + 28y = -40
   43y = -64

   Solve for y.
   43y = -64
   y = \frac{-64}{43}

   Substitute y = \frac{-64}{43} into the first equation and solve for x.
   4x + 5y = 8
   4x + 5\left(\frac{-64}{43}\right) = -8
   4x = \frac{-320}{43} = -8

   x = \frac{-24}{43}

   The solution to the system of equation is \left(\frac{-6}{43}, \frac{-64}{43}\right).
2-3 The Remainder and Factor Theorems

80. **SAT/ACT** In the figure, an equilateral triangle is drawn with an altitude that is also the diameter of the circle. If the perimeter of the triangle is 36, what is the circumference of the circle?

![Equilateral Triangle with Altitude as Diameter](image)

A. $6\sqrt{2} \pi$
B. $6\sqrt{3} \pi$
C. $12\sqrt{2} \pi$
D. $12\sqrt{3} \pi$
E. $36 \pi$

**SOLUTION:**
If the perimeter of the equilateral triangle is 36, then each side of the triangle measures 12. Also, each angle measures 60°. We can analyze half of the equilateral triangle, using the diameter of the circle as one of the legs.

![Diagram of half equilateral triangle with radius](image)

Since, this triangle is a 30-60-90 right triangle, $x = 6\sqrt{3}$. $x$ is also the diameter of the circle. The circumference of a circle is $C = \pi d$. So, the circumference of the circle is $C = \pi (6\sqrt{3})$ or $6\sqrt{3} \pi$.

The correct answer is B.

81. **REVIEW** If $(3, -7)$ is the center of a circle and $(8, 5)$ is on the circle, what is the circumference of the circle?

F. $13 \pi$
G. $15 \pi$
H. $18 \pi$
J. $25 \pi$
K. $26 \pi$

**SOLUTION:**
The distance from the center of a circle to a point on the circle is equal to the radius of the circle. Use the distance formula and the two points to find the radius of the circle.

$r = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$

$r = \sqrt{(8 - 3)^2 + [5 - (-7)]^2}$

$r = \sqrt{(5)^2 + (12)^2}$

$r = \sqrt{169}$

$r = 13$

The circumference of a circle is $C = \pi d$ or $2\pi r$. So, the circumference of the circle is $C = 2\pi (13)$ or $26 \pi$.

The correct answer is K.
2-3 The Remainder and Factor Theorems

82. REVIEW The first term in a sequence is \( x \). Each subsequent term is three less than twice the preceding term. What is the 5th term in the sequence?

A 8\( x \) – 21
B 8\( x \) – 15
C 16\( x \) – 39
D 16\( x \) – 45
E 32\( x \) – 93

**SOLUTION:**

\[
\begin{array}{c|c}
\text{1}^{\text{st}} \text{ term} & x \\
\text{2}^{\text{nd}} \text{ term} & 2(x) - 3 = 2x - 3 \\
\text{3}^{\text{rd}} \text{ term} & 2(2x - 3) - 3 = 4x - 9 \\
\text{4}^{\text{th}} \text{ term} & 2(4x - 9) - 3 = 8x - 21 \\
\text{5}^{\text{th}} \text{ term} & 2(8x - 21) - 3 = 16x - 45 \\
\end{array}
\]

The correct answer is D.

83. Use the graph of the polynomial function. Which is not a factor of \( x^5 + x^4 - 3x^3 - 3x^2 - 4x - 4 \)?

F \((x - 2)\)
G \((x + 2)\)
H \((x - 1)\)
J \((x + 1)\)

**SOLUTION:**

The graph suggests that –2, –1, and 2 are zeros of the function. Thus, \((x + 2)\), \((x + 1)\), and \((x - 2)\) are factors of \( f(x) \).

The correct answer is H.