16. **POPULATION** In a particular county, the population of the two largest cities can be modeled by \( f(x) = 200x + 25 \) and \( g(x) = 175x - 15 \), where \( x \) is the number of years since 2000 and the population is in thousands.

a. What is the population of the two cities combined after any number of years?

b. What is the difference in the populations of the two cities?

**SOLUTION:**

a. The population of the cities after \( x \) years is the sum of the individual populations.

\[
(f + g)(x) = f(x) + g(x)
\]

\[
= 200x + 25 + 175x - 15
\]

\[
= 375x + 10
\]

b. The difference in the populations of the cities is given by:

\[
(f - g)(x) = 200x + 25 - (175x - 15)
\]

\[
= 25x + 40
\]
For each pair of functions, find \( f \circ g \) and \( g \circ f \), if they exist. State the domain and range for each composed function.

21. \( f = \{(-15, -5), (-4, 12), (1, 7), (3, 9)\} \) 
   \( g = \{(3, -9), (7, 2), (8, -6), (12, 0)\} \)

**SOLUTION:**
The range of \( g(x) \) is not a subset of the domain of \( f(x) \).
So, \( f \circ g \) is undefined.
\[
\begin{align*}
[g \circ f](x) &= g[f(x)] \\
[g \circ f](-4) &= g[f(-4)] \\
&= g(12) \\
&= 0 \\
[g \circ f](1) &= g[f(1)] \\
&= g(7) \\
&= 2 \\
g \circ f &= \{(-4,0), (1,2)\}
\end{align*}
\]

22. \( f = \{(-1, 11), (2, -2), (5, -7), (4, -4)\} \) 
   \( g = \{(5, -4), (4, -3), (-1, 2), (2, 3)\} \)

**SOLUTION:**
\[
\begin{align*}
[f \circ g](x) &= f[g(x)] \\
[f \circ g](-1) &= f[g(-1)] \\
&= f(2) \\
&= -2 \\
f \circ g &= \{(-1,-2)\}
\end{align*}
\]
The range of \( f(x) \) is not a subset of the domain of \( g(x) \).
So, \( g \circ f \) is undefined.
31. \( f(x) = x^2 + 6x - 2 \)
   \( g(x) = x - 6 \)

**SOLUTION:**
\[
[f \circ g](x) = f[g(x)] = f(x - 6) = (x - 6)^2 + 6(x - 6) - 2 = x^2 - 6x - 2
\]
\[
[g \circ f](x) = g[f(x)] = g(x^2 + 6x - 2) = x^2 + 6x - 2 - 6 = x^2 + 6x - 8
\]

**For** \([f \circ g](x)\), \(D = \{\text{all real numbers}\}\), \(R = \{y \mid y \geq -11\}\)

**For** \([g \circ f](x)\), \(D = \{\text{all real numbers}\}\), \(R = \{y \mid y \geq -17\}\)

33. \( f(x) = 4x - 1 \)
   \( g(x) = x^3 + 2 \)

**SOLUTION:**
\[
[f \circ g](x) = f[g(x)] = f(x^3 + 2) = 4(x^3 + 2) - 1 = 4x^3 + 7
\]
\[
[g \circ f](x) = g[f(x)] = g(4x - 1) = (4x - 1)^3 + 2 = 64x^3 - 48x^2 + 12x + 1
\]

**For** \([f \circ g](x)\), \(D = \{\text{all real numbers}\}\), \(R = \{\text{all real numbers}\}\)

**For** \([g \circ f](x)\), \(D = \{\text{all real numbers}\}\), \(R = \{\text{all real numbers}\}\)
36. **FINANCE** A ceramics store manufactures and sells coffee mugs. The revenue \( r(x) \) from the sale of \( x \) coffee mugs is given by \( r(x) = 6.5x \). Suppose the function for the cost of manufacturing \( x \) coffee mugs is \( c(x) = 0.75x + 1850 \).

a. Write the profit function.

b. Find the profit on 500, 1000, and 5000 coffee mugs.

**SOLUTION:**

a. The profit function \( P(x) \) is given by 
\[ P(x) = r(x) - c(x) \] 
where \( r(x) \) is the revenue function and \( c(x) \) is the cost function. 
So: 
\[ P(x) = 6.5x - (0.75x + 1850) = 5.75x - 1850 \]

b. 
\[ P(500) = 5.75(500) - 1850 = 1025 \] 
\[ P(1000) = 5.75(1000) - 1850 = 5750 - 1850 = 3900 \] 
\[ P(5000) = 5.75(5000) - 1850 = 26900 \]

Perform each operation if \( f(x) = x^2 + x - 12 \) and 
\( g(x) = x - 3 \). State the domain of the resulting function.

39. \( 2(g \cdot f)(x) \)

**SOLUTION:**
\[ 2(g \cdot f)(x) = 2 \cdot g(x) \cdot f(x) = 2(x - 3)(x^2 + x - 12) = 2x^3 - 4x^2 - 30x + 72 \] 
D = {all real numbers}

45. \( f[h(-3)] \)

**SOLUTION:**
\[ f[h(x)] = f[x^2 + 6x + 8] = 5(x^2 + 6x + 8) = 5x^2 + 30x + 40 \]
Substitute \( x = -3 \).
\[ f[h(-3)] = 5(-3)^2 + 30(-3) + 40 = 45 - 90 + 40 = -5 \]

47. \( f[g(3a)] \)

**SOLUTION:**
\[ f[g(3a)] = f[-6a + 1] = 5(-6a + 1) = -30a + 5 \]

49. \( g(f(a^2 - a)) \)

**SOLUTION:**
\[ g[f(a^2 - a)] = g[5(a^2 - a)] = 5(5a^2 - 5a) = -2(5a^2 - 5a) + 1 = -10a^2 + 10a + 1 \]
If \( f(x) = x + 2 \), \( g(x) = -4x + 3 \), and \( h(x) = x^2 - 2x + 1 \), find each value.

53. \( [(f + g) \cdot h](1) \)

**SOLUTION:**
\[
[(f + g) \cdot h](x) = [f + g](x) \cdot h(x) \\
= [f(x) + g(x)] \cdot h(x) \\
= f(x) \cdot h(x) + g(x) \cdot h(x)
\]
Substitute \( x = 1 \).
\[
[(f + g) \cdot h](1) = f(1) \cdot h(1) + g(1) \cdot h(1) \\
= (3)(0) + (-1)(0) \\
= 0
\]

56. \( g \circ (h \cdot f)(-4) \)

**SOLUTION:**
\[
g \circ (h \cdot f)(-4) = g(h[f(-4)]) \\
= g(h(-2)) \\
= g(9) \\
= -36 + 3 \\
= -33
\]

60. **CCSS CRITIQUE** Chris and Tobias are finding the composition \( (f \circ g)(x) \) where \( f(x) = x^2 + 2x - 8 \) and \( g(x) = x^2 + 8 \). Is either of them correct? Explain your reasoning.

**Chris**
\[
(f \circ g)(x) = f[g(x)] \\
= (x^2 + 8)^2 + 2(x^2 + 8) - 8 \\
= x^4 + 16x^2 + 64 + 2x^2 + 16 - 8 \\
= x^4 + 16x^2 + 2x^2 + 58
\]

**Tobias**
\[
(f \circ g)(x) = f[g(x)] \\
= (x^2 + 8)^2 + 2(x^2 + 8) - 8 \\
= x^4 + 16x^2 + 64 + 2x^2 + 16 - 8 \\
= x^4 + 18x^2 + 72
\]

**SOLUTION:**
Tobias is correct. Chris did not substitute \( g(x) \) for every \( x \) in \( f(x) \).

61. **CHALLENGE** Given \( f(x) = \sqrt{x^2} \) and \( g(x) = \sqrt{x^6} \), determine the domain for each of the following.

a. \( g(x) \cdot g(x) \)

**SOLUTION:**
\[
g(x) = \sqrt{x^6} \cdot \sqrt{x^6} = x^6 \\
D = \{ \text{all real numbers} \}
\]

b. \( f(x) \cdot f(x) \)

**SOLUTION:**
\[
f(x) = \sqrt{x^3} \cdot \sqrt{x^3} = x^3 \\
\text{Since } f(x) \text{ is defined for } x \geq 0, \text{ the domain of } f(x) \cdot f(x) \text{ is } \{x | x \geq 0\}.
\]